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B.Sc. Part-I (Hons), Paper-I (Trigonometry)

Resolution into factors (Some problems)

(1) Prove that $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)$ --- to $\infty = \frac{2}{\pi}$

Proof:- we know that

$$\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right) \text{ --- to inf.}$$

Putting $\theta = \frac{\pi}{2} \Rightarrow \frac{\theta}{\pi} = \frac{1}{2}$

Then,

$$\sin\frac{\pi}{2} = \frac{\pi}{2}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{2^2.2^2}\right)\left(1-\frac{1}{3^2.2^2}\right) \text{ --- to inf.}$$

$$\Rightarrow 1 = \frac{\pi}{2}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right) \text{ --- to inf.}$$

$$\Rightarrow \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right) \text{ --- to } \infty = \frac{2}{\pi}$$; proved

(2) Prove that $\frac{1}{3^4}+\frac{3}{5^4}+\frac{6}{7^4}+\frac{10}{9^4}+$ --- to $\infty = \frac{\pi^2}{64}\left(1-\frac{\pi^2}{12}\right)$

Proof :- The give series may be written as

$$\frac{1}{8}\frac{(3^2-1)}{3^4}+\frac{1}{8}\frac{(5^2-1)}{5^4}+\frac{1}{8}\frac{(7^2-1)}{7^4}+\frac{1}{8}\frac{(9^2-1)}{9^4}+\text{---}$$

$$=\frac{1}{8}\left[\frac{3^2}{3^4}-\frac{1}{3^4}+\frac{5^2}{5^4}-\frac{1}{5^4}+\frac{7^2}{7^4}-\frac{1}{7^4}+\frac{9^2}{9^4}-\frac{1}{9^4}+\text{---}\right]$$

$$=\frac{1}{8}\left[\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\text{---}\right)-\left(\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\text{---}\right)\right]$$

$$=\frac{1}{8}\left[\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\text{---}\right)-\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\text{---}\right)\right]$$

$$=\frac{1}{8}\left[\frac{\pi^2}{8}-\frac{\pi^4}{96}\right]=\frac{1}{8}\frac{\pi^2}{8}\left[1-\frac{\pi^2}{12}\right]$$

$$=\frac{\pi^2}{64}\left[1-\frac{\pi^2}{12}\right]$$; proved

(2)

(3) Prove that $\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\cdots = \frac{\pi}{4}$

Proof:- we have already know that

$$\cos x = \left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{3^2\pi^2}\right)\left(1-\frac{4x^2}{5^2\pi^2}\right)\left(1-\frac{4x^2}{7^2\pi^2}\right)\cdots$$

$$\Rightarrow \frac{\cos x}{1-\frac{4x^2}{\pi^2}} = \left(1-\frac{4x^2}{3^2\pi^2}\right)\left(1-\frac{4x^2}{5^2\pi^2}\right)\left(1-\frac{4x^2}{7^2\pi^2}\right)\cdots$$

Putting $x=\frac{\pi}{2} \Rightarrow \frac{2x}{\pi}=1$, we get

$$\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{1-\frac{4x^2}{\pi^2}} = \left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\cdots$$

$$\text{L.H.S} = \lim_{x\to\frac{\pi}{2}} \frac{\frac{d}{dx}(\cos x)}{\frac{d}{dx}\left(1-\frac{4x^2}{\pi^2}\right)} \quad \text{[By L' Hospital Rule]}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{-\sin x}{-\frac{8x}{\pi^2}} = \frac{\sin\frac{\pi}{2}}{\frac{\cancel{8}4\frac{\pi}{2}}{\pi^2}} = \frac{\cancel{\pi^2}\sin\frac{\pi}{2}}{4\cancel{\pi}} = \frac{\pi}{4}$$

Hence, $\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\cdots = \frac{\pi}{4}$; proved.

(4) Prove that $\frac{\sinh\pi}{\pi} = \left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\cdots$ to inf

Proof:- we Know that

$$\sinh\theta = \theta\left(1+\frac{\theta^2}{\pi^2}\right)\left(1+\frac{\theta^2}{2^2\pi^2}\right)\left(1+\frac{\theta^2}{3^2\pi^2}\right)\cdots \text{ to inf}$$

Putting $\theta=\pi$

$$\sinh\pi = \pi\left(1+\frac{\cancel{\pi^2}}{\cancel{\pi^2}}\right)\left(1+\frac{\pi^2}{2^2\pi^2}\right)\left(1+\frac{\pi^2}{3^2\pi^2}\right)\cdots \text{ to inf}$$

$$= \pi\left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\cdots \text{ to inf}$$

$$\therefore \frac{\sinh\pi}{\pi} = \left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\cdots \text{ to inf}$$

; proved.